1. Square roots and signs

Diagnostic question

For $x < 3$, simplify $\dfrac{\sqrt{(x-3)^2}}{x-3}$.

Hint. Use $\sqrt{u^2} = |u|$, then decide the sign of $x - 3$.

Worked solution

The square root is non-negative, so $\sqrt{(x-3)^2} = |x-3|$. Since $x < 3$, we have $x - 3 < 0$ and therefore $|x-3| = -(x-3)$.

Hence

$$\frac{\sqrt{(x-3)^2}}{x-3} = \frac{-(x-3)}{x-3} = -1.$$

The denominator is non-zero because $x < 3$.

Answer: -1

What to notice

Writing $\sqrt{u^2} = u$ loses the sign information when $u < 0$.

Matched follow-up

For $x > 4$, simplify $\dfrac{\sqrt{(4-x)^2}}{x-4}$.

Since $x > 4$, $4 - x < 0$, so $\sqrt{(4-x)^2} = |4-x| = x - 4$. Dividing by the non-zero quantity $x - 4$ gives 1.

Follow-up answer: 1

Teaching note

Ask the student to explain the step that made the difference, then try the follow-up without looking at the worked solution.

Further practice: Modulus and graphs; Pure Year 2: Functions and Graphs.

2. Factorisation and domain

Diagnostic question

Simplify $\dfrac{x^3+27}{x^2-9}$, stating all excluded values of x.

Hint. Record the zeros of the original denominator before using the sum-of-cubes identity.

Worked solution

The original denominator is zero at $x=3$ and $x=-3$, so both values are excluded.

Factor the numerator and denominator:

$$x^3+27=(x+3)(x^2-3x+9), \qquad x^2-9=(x+3)(x-3).$$

For permitted values, cancel the common factor $x+3$. This gives $\dfrac{x^2-3x+9}{x-3}$, with $x\neq -3, 3$. The cancellation does not restore $x=-3$ to the original domain.

Answer: $\dfrac{x^2-3x+9}{x-3}$, with $x\neq \pm 3$

What to notice

A cancelled factor can remove a visible restriction from the formula, but it does not change the domain of the original expression.

Matched follow-up

Simplify $\dfrac{x^3-125}{x^2-25}$, stating all excluded values.

The original denominator gives $x\neq \pm 5$. Factor $x^3-125=(x-5)(x^2+5x+25)$ and $x^2-25=(x-5)(x+5)$.

Cancel $x-5$ for permitted values to obtain $\dfrac{x^2+5x+25}{x+5}$, still with $x\neq \pm 5$.

Follow-up answer: $\dfrac{x^2+5x+25}{x+5}$, with $x\neq \pm 5$

Teaching note

Ask the student to explain the step that made the difference, then try the follow-up without looking at the worked solution.

Further practice: Factorisation and useful identities; Pure Year 1: Algebraic Methods.

3. Rational inequalities

Diagnostic question

Solve $\dfrac{x-1}{x+2} \geq 2$.

Hint. Move everything to one side and use a sign chart. Do not multiply by an expression whose sign you have not fixed.

Worked solution

First exclude $x = -2$. Subtracting 2 gives

$$\frac{x-1}{x+2} - 2 = \frac{-x-5}{x+2} \geq 0.$$

The critical values are -5 and -2. For $x < -5$, the numerator is positive and the denominator negative. For $-5 < x < -2$, both are negative. For $x > -2$, the numerator is negative and the denominator positive.

The fraction is non-negative on $[-5, -2)$. Include -5, where the numerator is zero; exclude -2, where the expression is undefined.

Answer: $-5 \leq x < -2$

What to notice

Multiplying by $x + 2$ without considering its sign can reverse an inequality incorrectly or lose an entire interval.

Matched follow-up

Solve $\dfrac{x+2}{x-1} \leq 3$.

With $x \neq 1$, rearrange to $\dfrac{5-2x}{x-1} \leq 0$. The critical values are 1 and 5/2.

The fraction is negative for $x < 1$, positive for $1 < x < 5/2$, and negative for $x > 5/2$. Include 5/2, where the fraction is zero, and exclude 1.

Follow-up answer: $x < 1$ or $x \geq \dfrac{5}{2}$

Teaching note

Ask the student to explain the step that made the difference, then try the follow-up without looking at the worked solution.

Further practice: Pure Year 1: Equations and Inequalities; Logic and proof.

4. Combining conditions

Diagnostic question

Find all real x such that $x^2 > 9$ and $x < 1$.

Hint. Solve each condition separately, then take their intersection.

Worked solution

The inequality $x^2 > 9$ gives $x < -3$ or $x > 3$. The second condition is $x < 1$.

Every value with $x < -3$ satisfies $x < 1$, while no value with $x > 3$ does. Therefore the intersection is $x < -3$.

Answer: $x < -3$

What to notice

The word "and" requires both conditions. Squaring inequalities can also hide a negative branch.

Matched follow-up

Find all real x such that $x^2 \leq 16$ and $x > -1$.

The first condition gives $-4 \leq x \leq 4$. Intersect this with $x > -1$ to obtain $-1 < x \leq 4$.

Follow-up answer: $-1 < x \leq 4$

Teaching note

Ask the student to explain the step that made the difference, then try the follow-up without looking at the worked solution.

Further practice: Logic and proof; Pure Year 1: Equations and Inequalities.

5. Modulus equations

Diagnostic question

Solve $|2x - 1| = x + 2$.

Hint. Split at $x = 1/2$, and keep the condition attached to each branch.

Worked solution

If $x \geq 1/2$, then $|2x - 1| = 2x - 1$. Thus $2x - 1 = x + 2$, giving $x = 3$, which satisfies this branch condition.

If $x < 1/2$, then $|2x - 1| = 1 - 2x$. Thus $1 - 2x = x + 2$, giving $x = -1/3$, which also satisfies its branch condition.

Both values satisfy the original equation, so the complete solution set is $\{-1/3, 3\}$.

Answer: $x = -\frac{1}{3}$ or $x = 3$

What to notice

Using both signs without checking branch conditions can admit values that do not satisfy the original equation.

Matched follow-up

Solve $|3x + 2| = x + 4$.

For $x \geq -2/3$, solve $3x + 2 = x + 4$, giving $x = 1$. For $x < -2/3$, solve $-3x - 2 = x + 4$, giving $x = -3/2$.

Both values meet their branch conditions and satisfy the original equation.

Follow-up answer: $x = 1$ or $x = -\frac{3}{2}$

Teaching note

Ask the student to explain the step that made the difference, then try the follow-up without looking at the worked solution.

Further practice: Modulus and graph reflections; Pure Year 2: Functions and Graphs.

6. Graphs and transformed inputs

Diagnostic question

The graph of a quadratic f opens upwards and has roots -2 and 4. Given $f(0) < 0$, find all x for which $f(2x-1) < 0$.

Hint. Find which inputs make f negative, then put $2x-1$ between those inputs.

Worked solution

An upward-opening quadratic is negative strictly between its roots, so $f(t) < 0$ exactly when $-2 < t < 4$.

Use $t = 2x - 1$:

$$-2 < 2x - 1 < 4 \quad \Longleftrightarrow \quad -1 < 2x < 5 \quad \Longleftrightarrow \quad -\frac{1}{2} < x < \frac{5}{2}.$$

The endpoints are excluded because the quadratic is zero there. The condition $f(0) < 0$ agrees with the sign of the graph between its roots.

Answer: $-\frac{1}{2} < x < \frac{5}{2}$

What to notice

Changing the input of a function requires solving for the new variable; the original root interval cannot simply be reused.

Matched follow-up

A quadratic g opens upwards and has roots 1 and 7. Find all x such that $g(3x+1) > 0$.

The function is positive outside its roots: $g(t) > 0$ when $t < 1$ or $t > 7$. Thus $3x + 1 < 1$ or $3x + 1 > 7$, giving $x < 0$ or $x > 2$.

Follow-up answer: $x < 0$ or $x > 2$

Teaching note

Ask the student to explain the step that made the difference, then try the follow-up without looking at the worked solution.

Further practice: Modulus and graphs; Pure Year 1: Graphs and Transformations.

7. Using identities efficiently

Diagnostic question

A positive real number x satisfies $x + 1/x = 4$. Find $(x - 1/x)^2$.

Hint. Compare the expansions of the sum squared and the difference squared.

Worked solution

Expanding gives $(x + 1/x)^2 = x^2 + 2 + 1/x^2$ and $(x - 1/x)^2 = x^2 - 2 + 1/x^2$.

The second expression is four less than the first. Hence

$$(x - 1/x)^2 = (x + 1/x)^2 - 4 = 16 - 4 = 12.$$

There is no need to solve a quadratic for x.

Answer: 12

What to notice

The middle term of a square matters: $(a - b)^2$ is not $a^2 - b^2$.

Matched follow-up

A positive real number x satisfies $x + 1/x = 5$. Find $(x - 1/x)^2$.

The same identity gives $(x - 1/x)^2 = (x + 1/x)^2 - 4 = 25 - 4 = 21$.

Follow-up answer: 21

Teaching note

Ask the student to explain the step that made the difference, then try the follow-up without looking at the worked solution.

Further practice: Factorisation and useful identities; Pure Year 1: Algebraic Expressions.

8. Counting distinct roots

Diagnostic question

For which values of $k > 0$ does $|x^2 - 4| = k$ have exactly two distinct real solutions?

Hint. Rewrite the equation as two equations for x^2, and check the point where one right-hand side becomes zero.

Worked solution

The equation is equivalent to $x^2 = 4 + k$ or $x^2 = 4 - k$. Since $k > 0$, the first equation always has two distinct real roots.

When $0 < k < 4$, the second equation also has two distinct non-zero roots. They differ from the first pair, so there are four roots in total.

When $k = 4$, the second equation gives only $x = 0$, so there are three distinct roots. When $k > 4$, it has no real roots, leaving exactly two.

Answer: $k > 4$

What to notice

At a transition value, the two expressions $+0$ and -0 are the same root. Count distinct roots separately at the boundary.

Matched follow-up

For which values of $k > 0$ does $|x^2 - 9| = k$ have exactly three distinct real solutions?

The equations are $x^2 = 9 + k$ and $x^2 = 9 - k$. For $0 < k < 9$ there are four distinct roots; for $k > 9$ there are two.

At $k = 9$, the roots are $x = 0$ and $x = \pm\sqrt{18} = \pm 3\sqrt{2}$. These are three distinct real roots, so $k = 9$ is the only value.

Follow-up answer: $k = 9$

Teaching note

Ask the student to explain the step that made the difference, then try the follow-up without looking at the worked solution.

Further practice: Modulus and graph reflections; Pure Year 2: Functions and Graphs.